Sparsity in Data Analysis and Computation

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Sparsity

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- emerging role over last $\sim$ 4 decades

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- emerging role over last $\sim$ 4 decades
- powerful tool
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  - for computation
- better understanding will have enormous impact

Data analysis

Data analysis: Images

Data analysis: Images

What is an image?

Data analysis: Images

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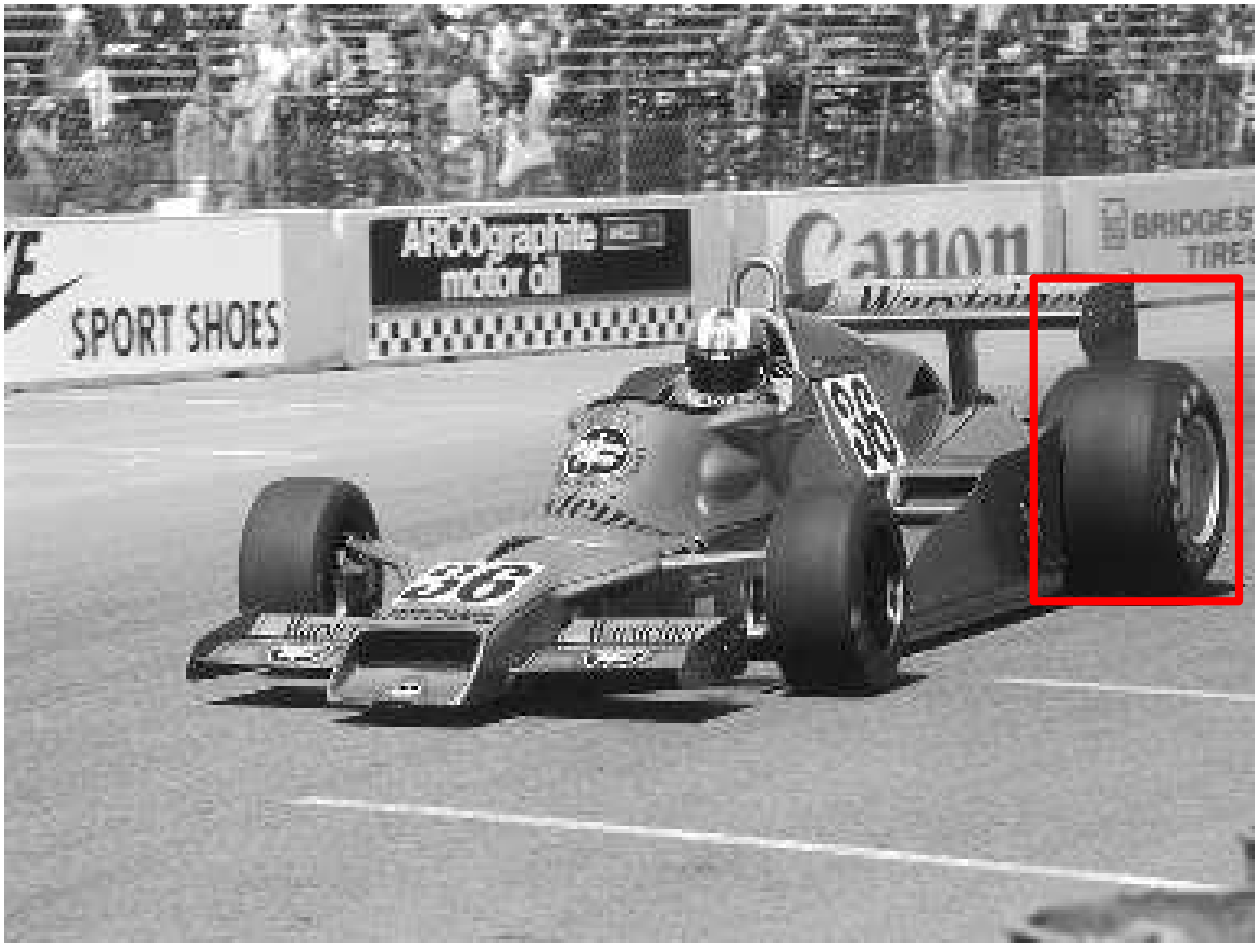

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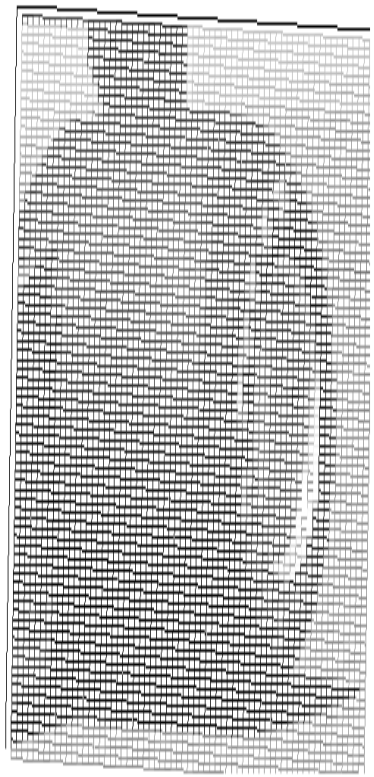

$$f : [a,b] \times [c,d] \longrightarrow \mathbb{R}_+$$

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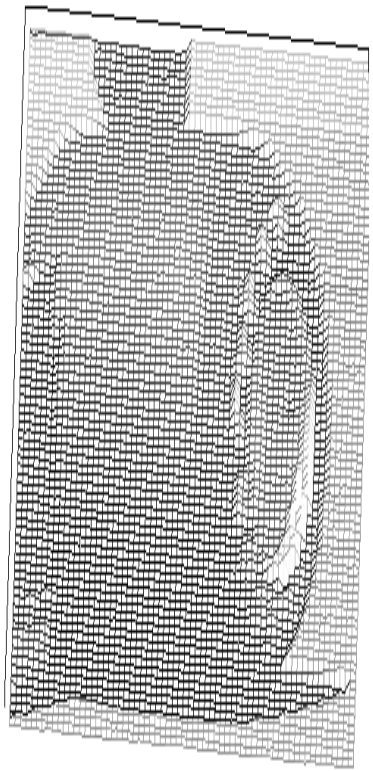

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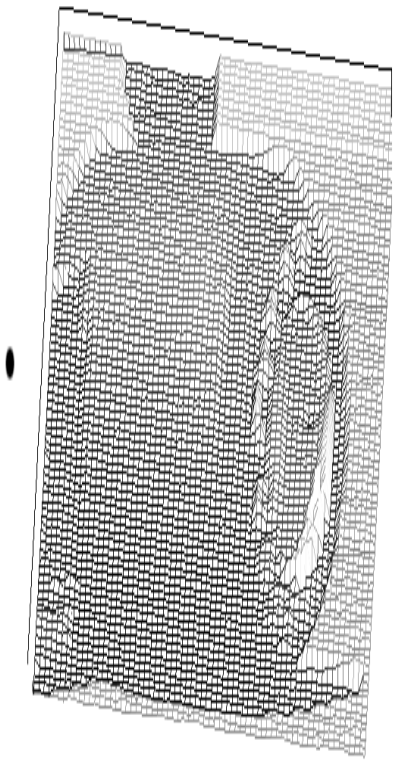

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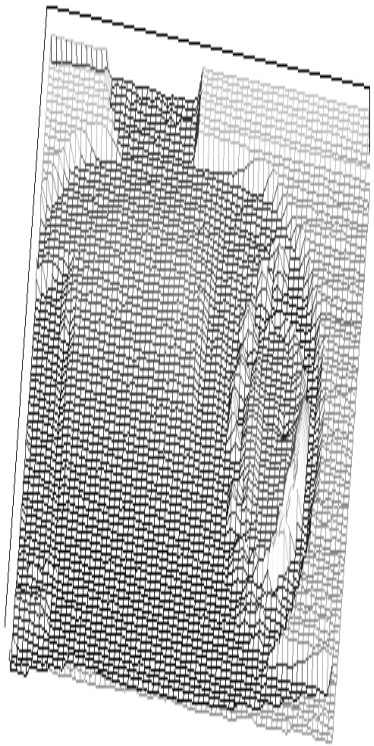

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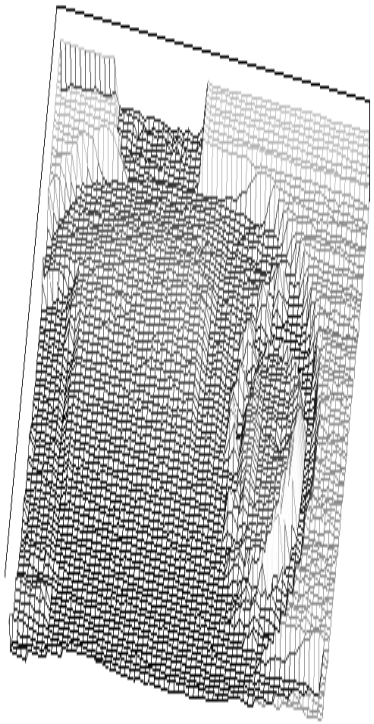

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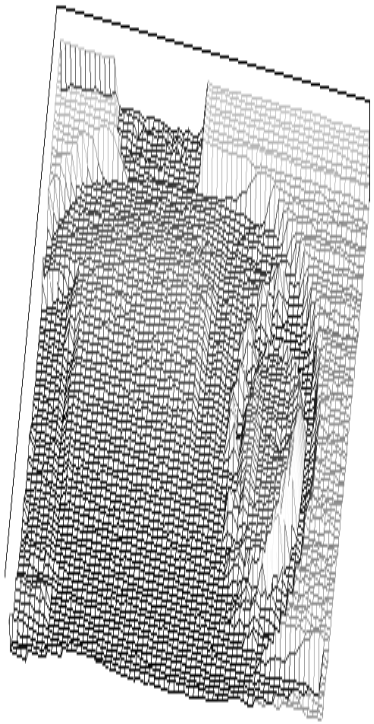

$$f : [a,b] \times [c,d] \longrightarrow [0,1]$$

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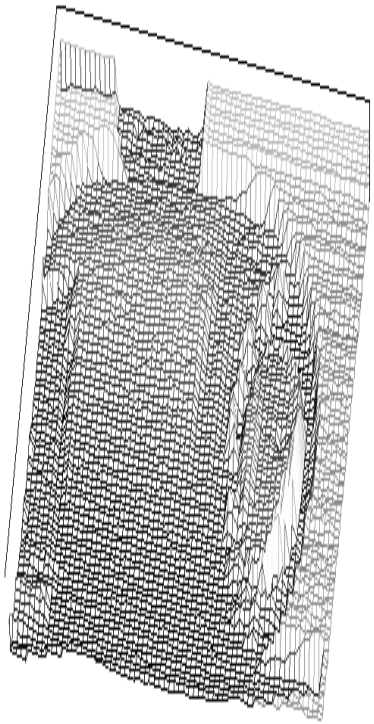

$$f : [a,b] \times [c,d] \longrightarrow [0,1]$$

sampled: $f\left(a + \frac{b-a}{M} m,\ c + \frac{d-c}{N} n\right)$

$m = 0, 1, \ldots, M$

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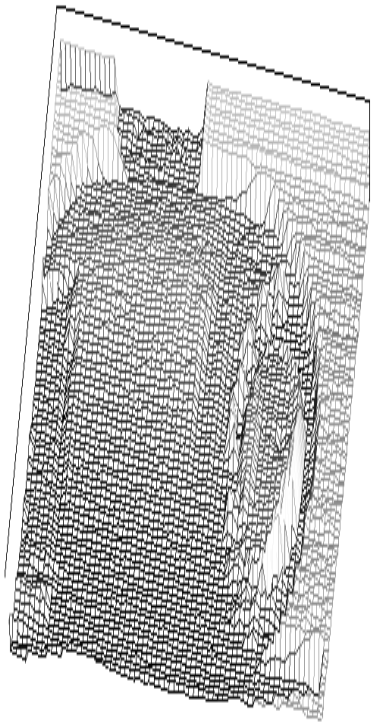

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or averaged:

$$\int_{|s - \frac{b-a}{M} m| < \frac{b-a}{2M}} \int_{|t - \frac{d-c}{N}| < \frac{d-c}{2N}} f(s,t)\, ds\, dt$$

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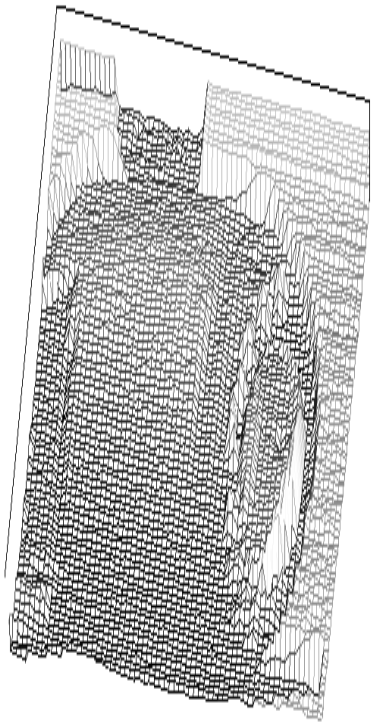

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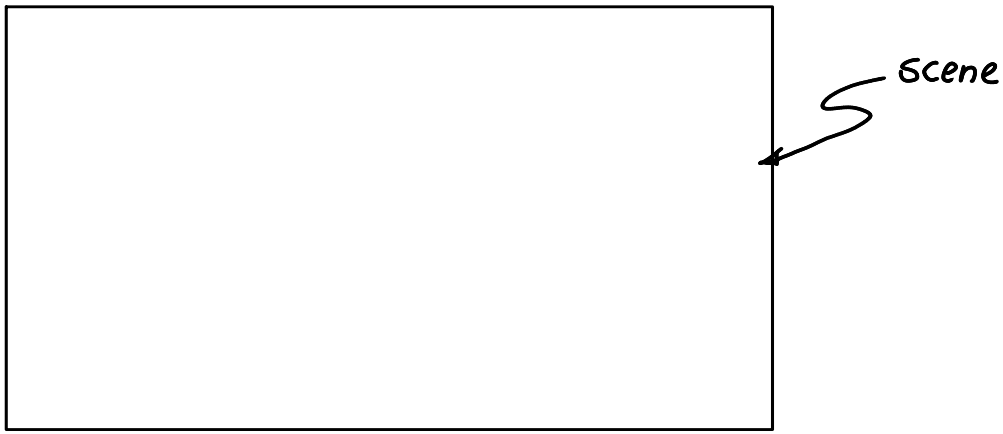

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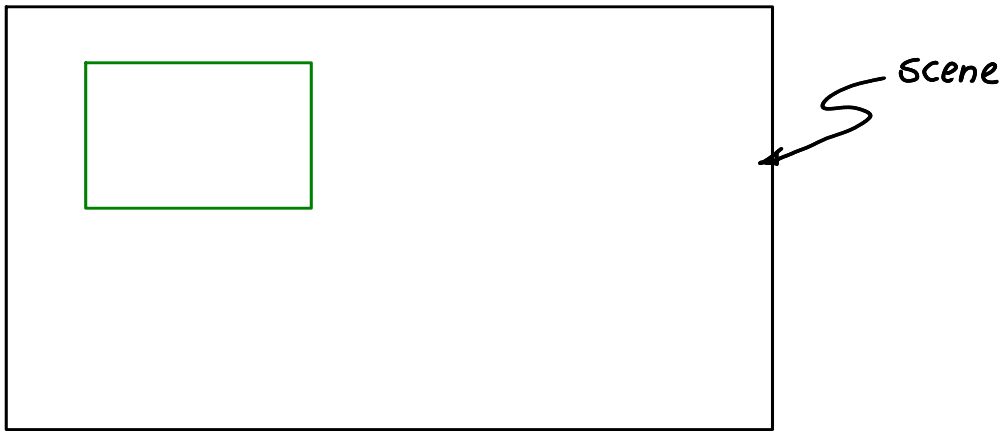

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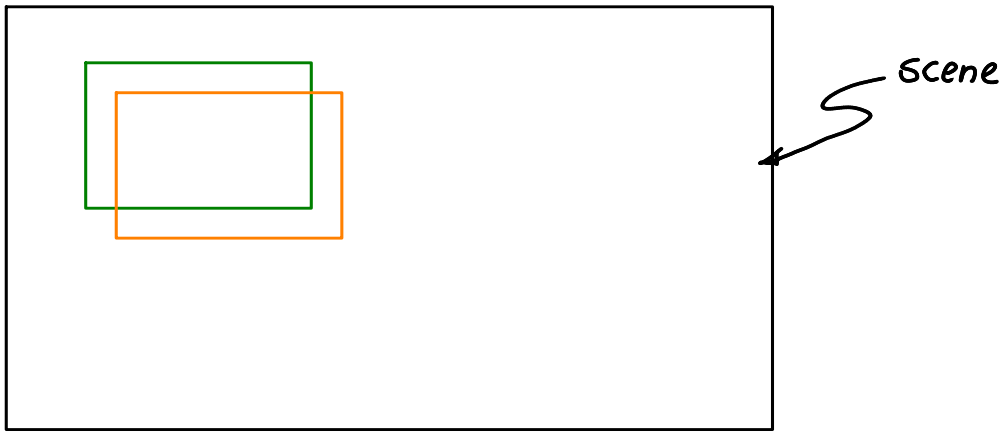

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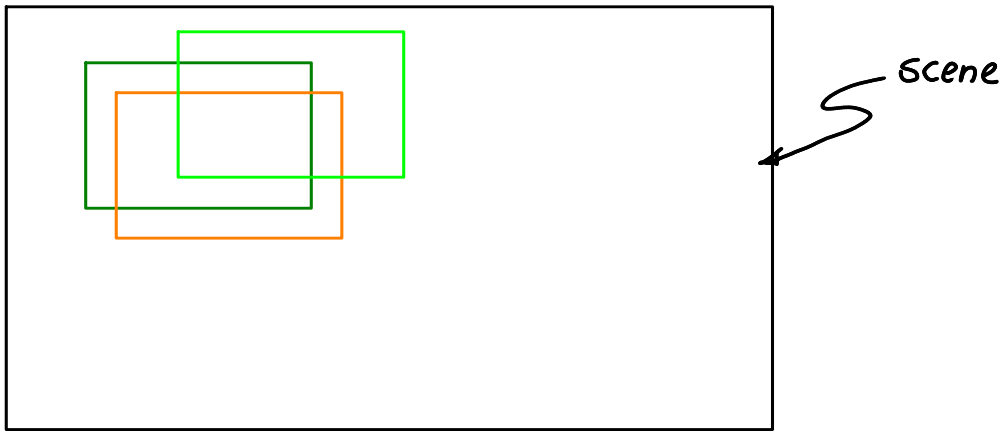

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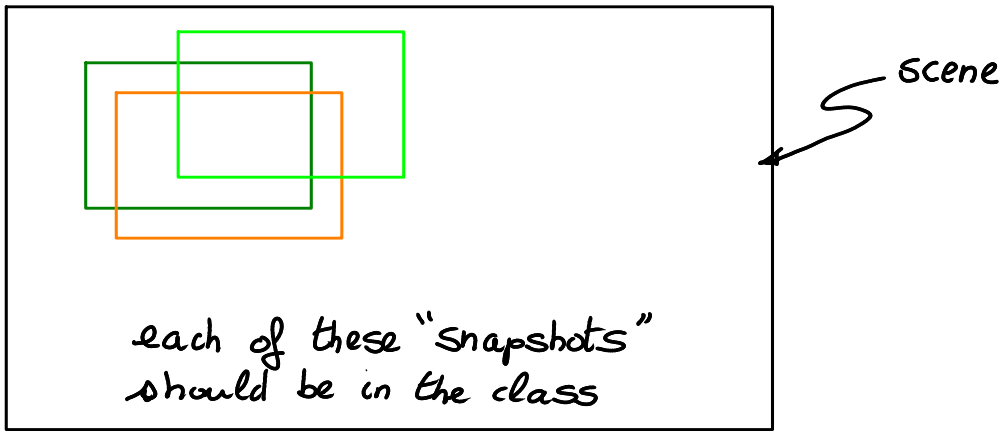

Data analysis: Images

Invariance under Translation Group

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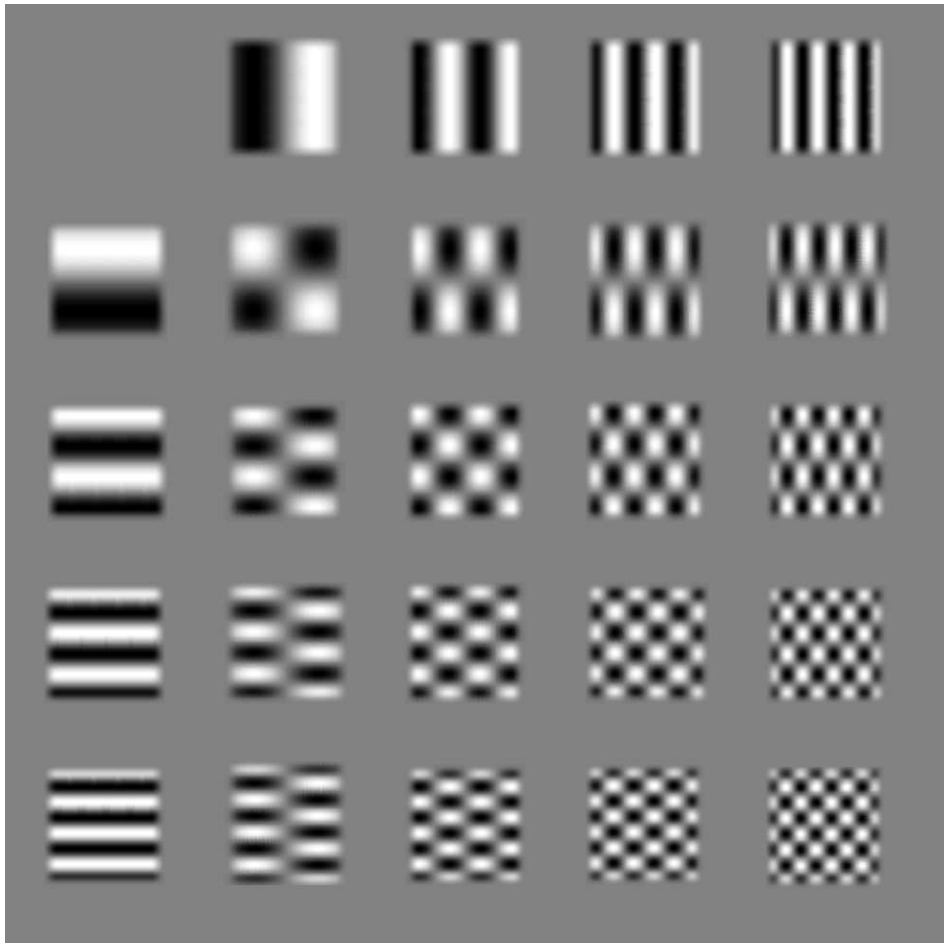

Data analysis: Images

JPEG standard :

uses DCT on 8×8 blocks

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technical reasons

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in early 80s: → 16x16

expected to go to even larger...

Data analysis: Images

In 1980s : start of use of wavelet transform for images.

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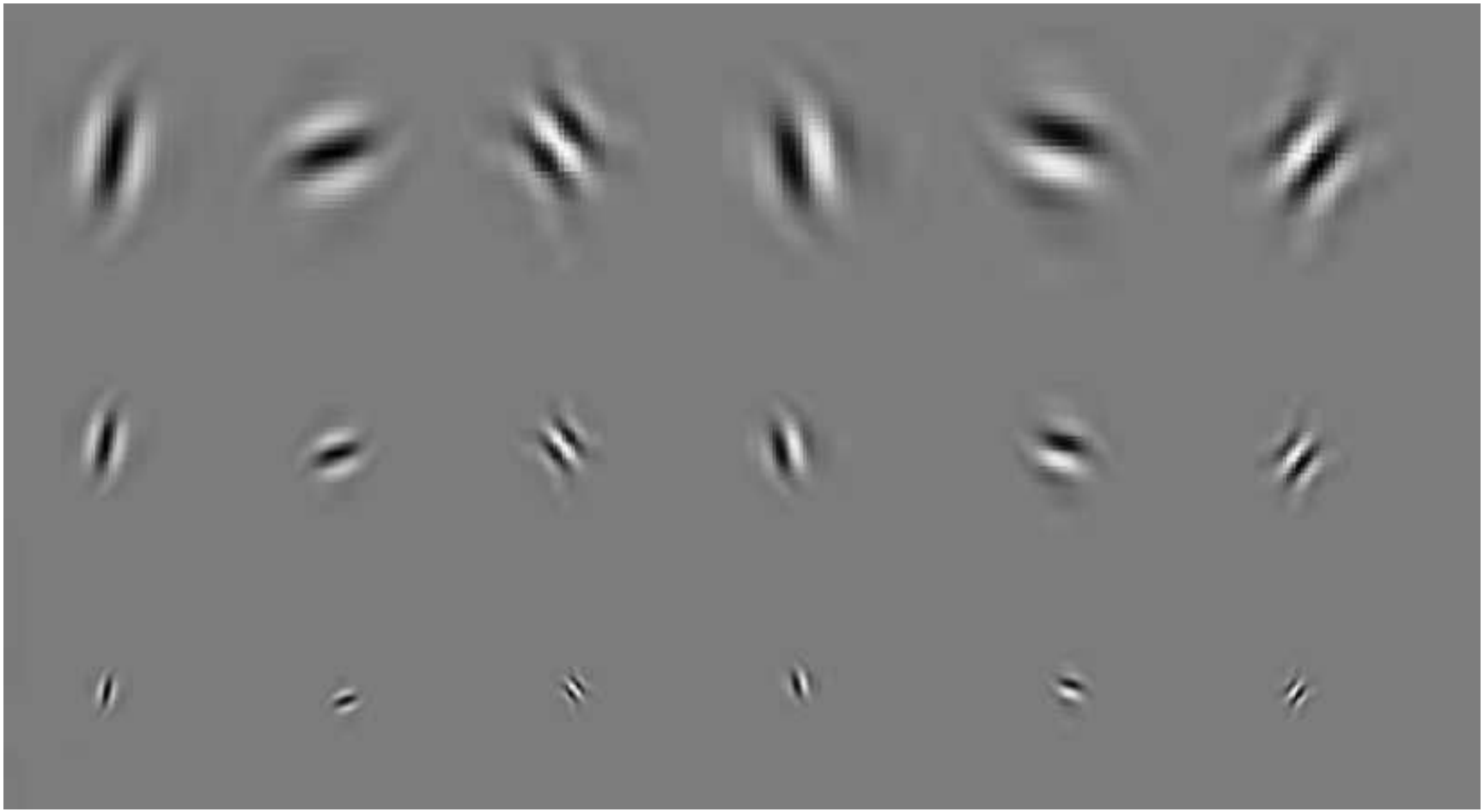
$\longrightarrow$ decomposition of images into different type of building blocks.

Data analysis: Images



Data analysis: Images

Wavelets

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⇒ need many more fine scale wavelets to cover the image domain than coarse scale wavelets

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⇒ need many more fine scale wavelets to cover the image domain than coarse scale wavelets

⇒ traditional representation of wavelet decompositions of an image.

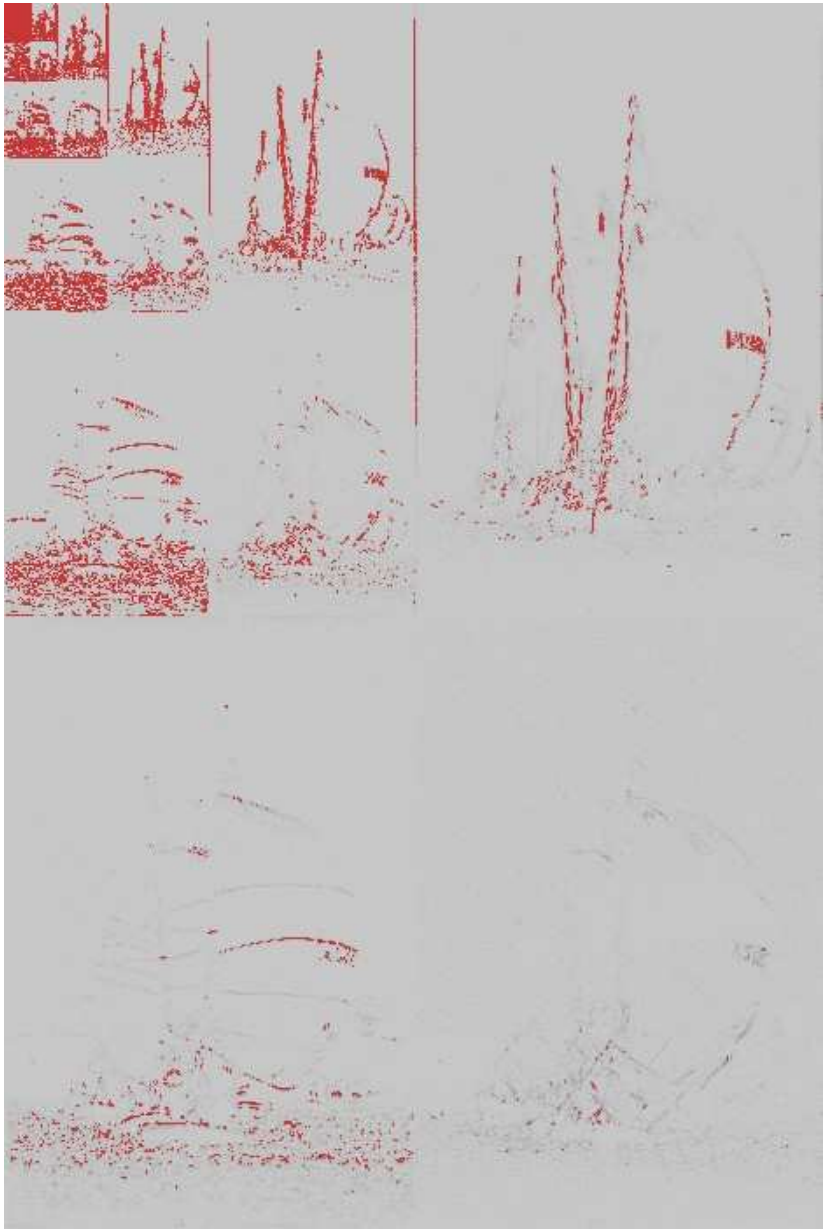

14255


14255

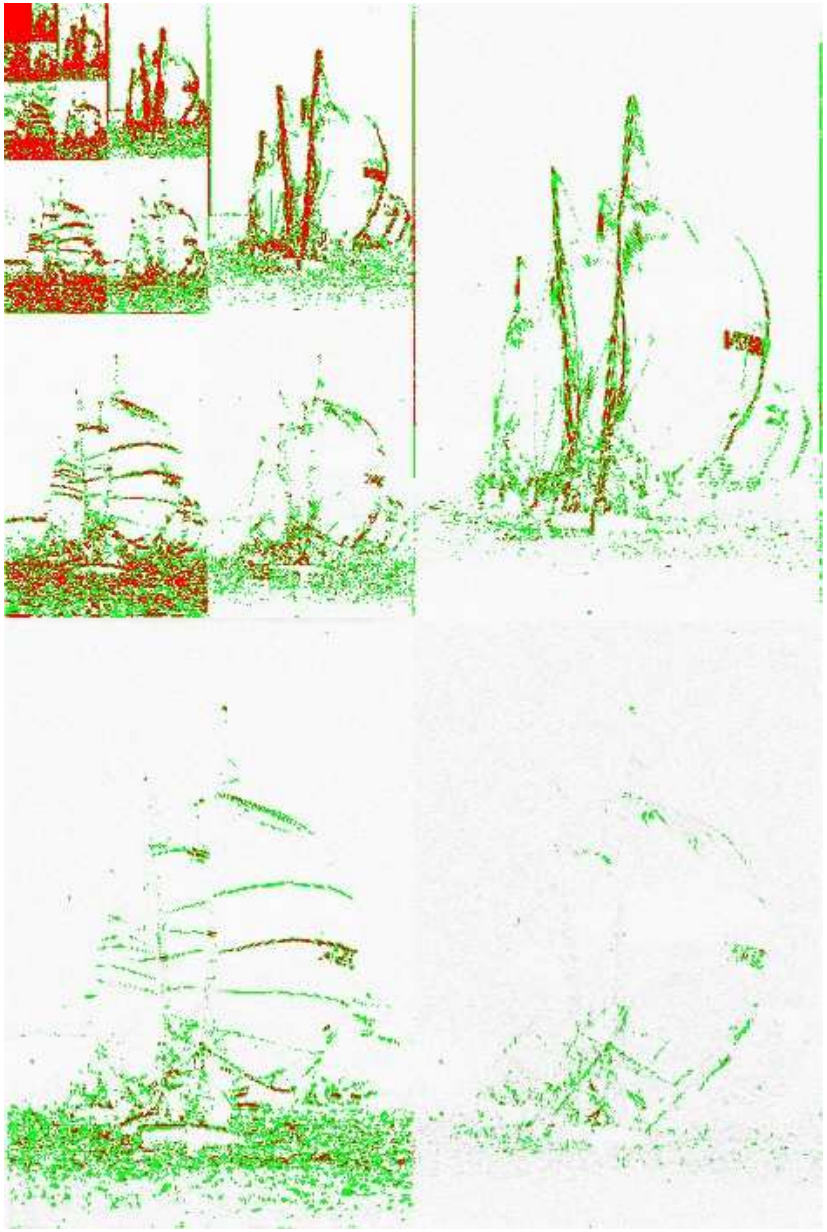

14255

Compression


14255



Compression Ratio: 3.3%



Compression Ratio: 10%

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In JPEG-2000 image standard:

wavelets instead of DCT.

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. digital movies, sports reporting

Data analysis: Images

Why are wavelets a good idea for images?

What was "wrong" with the Fourier analysis argument?

Really the difference between

Linear and Non-linear

approximation.

Data analysis: Images

Consider a simple class of functions on $\mathbb{T}$

$\mathcal{C}$ $\quad f \in \mathcal{C}$

$f : \mathbb{T} \longrightarrow \mathbb{C}$ "nice"

on $\mathcal{C}$: probability measure

invariant under translations

Then one can prove that the "best" basis in which the $f \in \mathcal{C}$ can be decomposed is the Fourier basis

Namely:

If one wants to find the basis $\varphi_1, \varphi_2, \dots \varphi_n, \dots$ of functions such that

$$\mathbb{E}\left(\int_{\mathbb{T}} \left| f(t) - \langle f, \varphi_1 \rangle \varphi_1(t) \right|^2 dt \right)$$

$$\vdots$$

$$\mathbb{E}\left(\int_{\mathbb{T}} \left| f(t) - \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n(t) \right|^2 dt \right)$$

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are minimal, then these must be the Fourier exponentials $e^{2\pi i n t}$

Data analysis: Images

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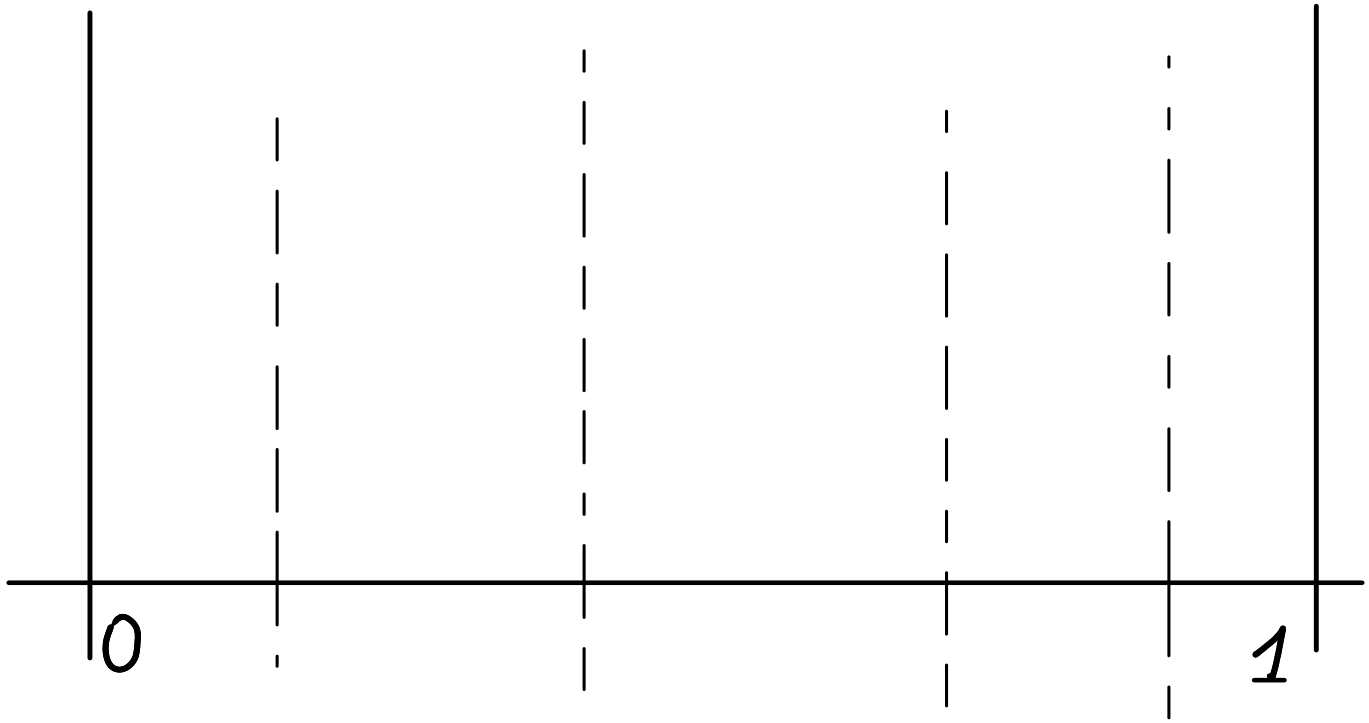

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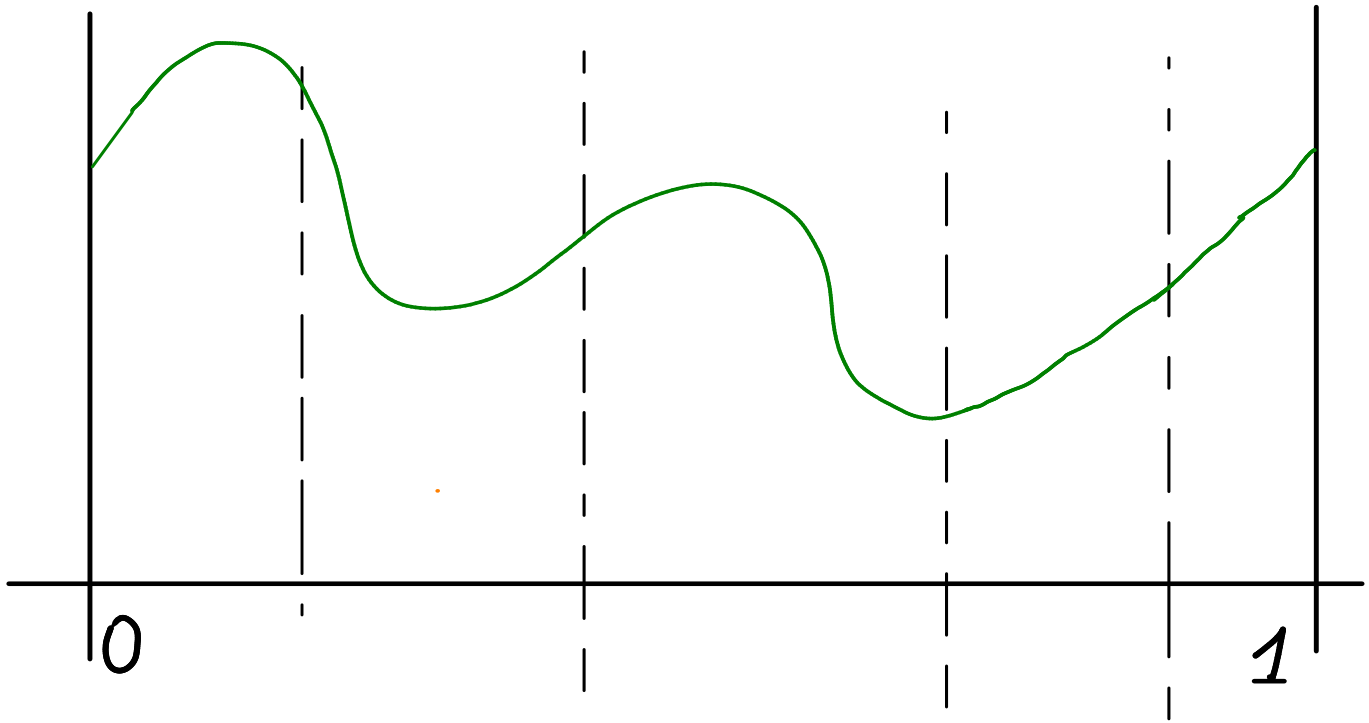

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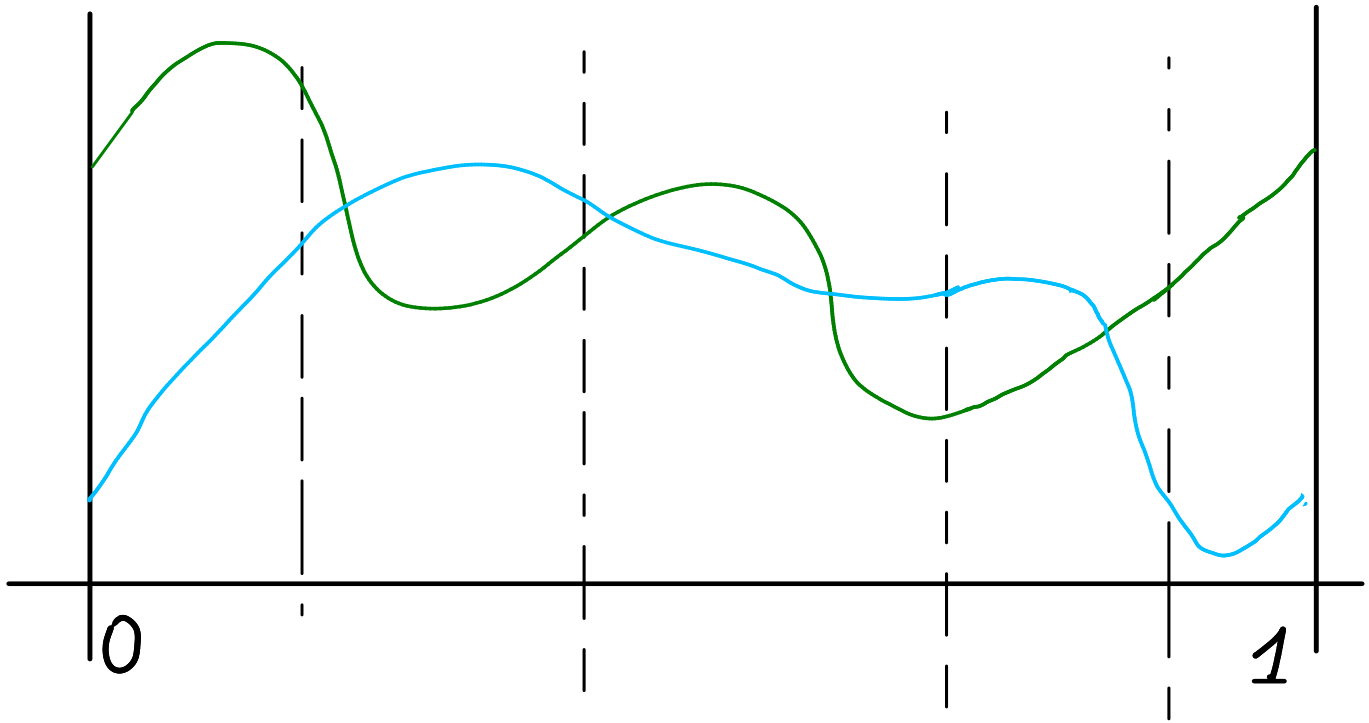

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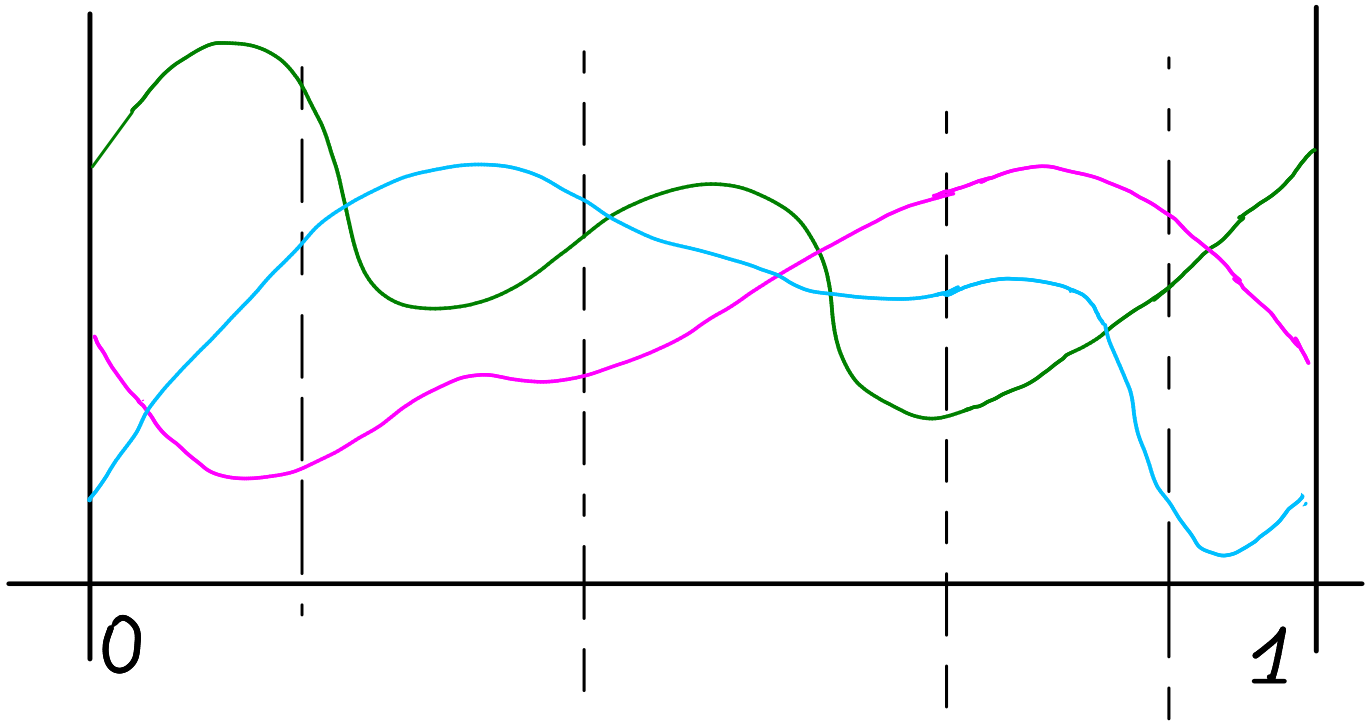

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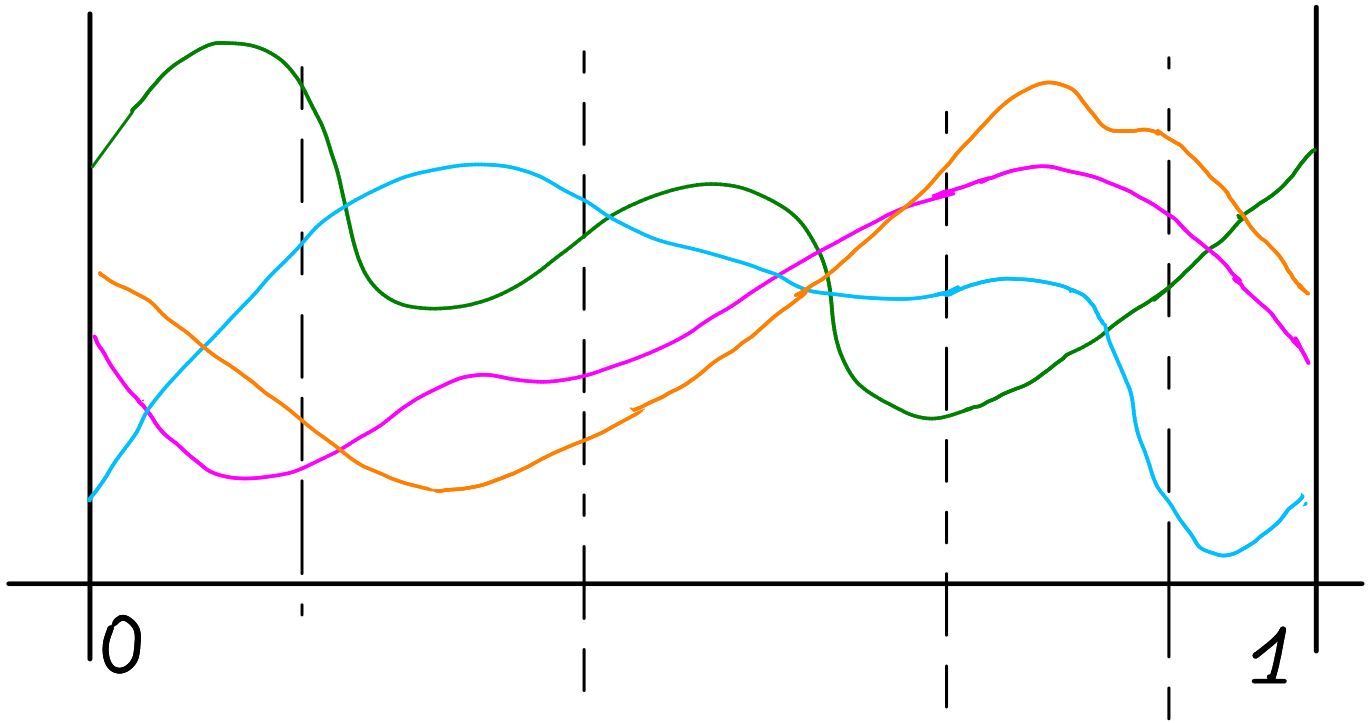

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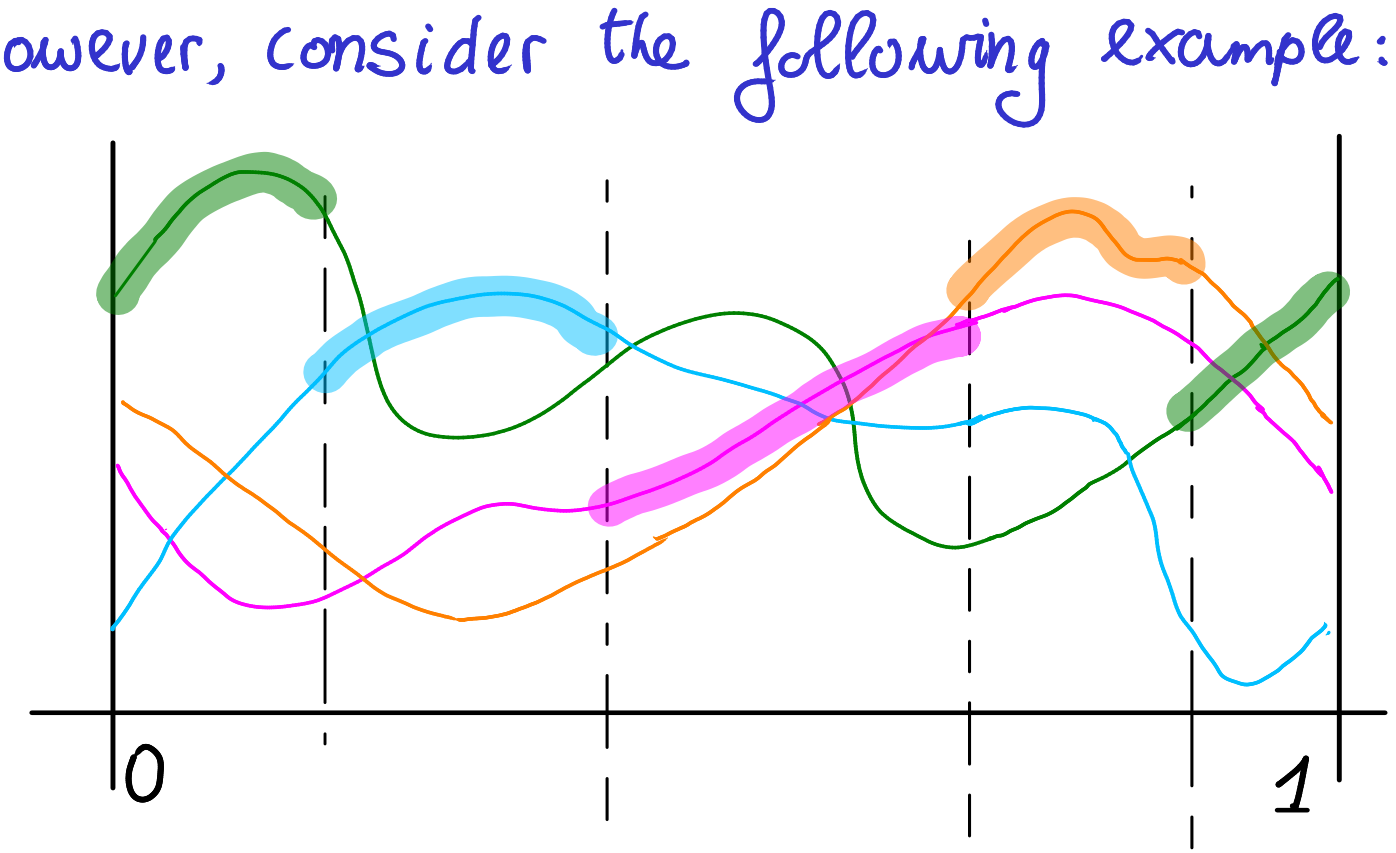

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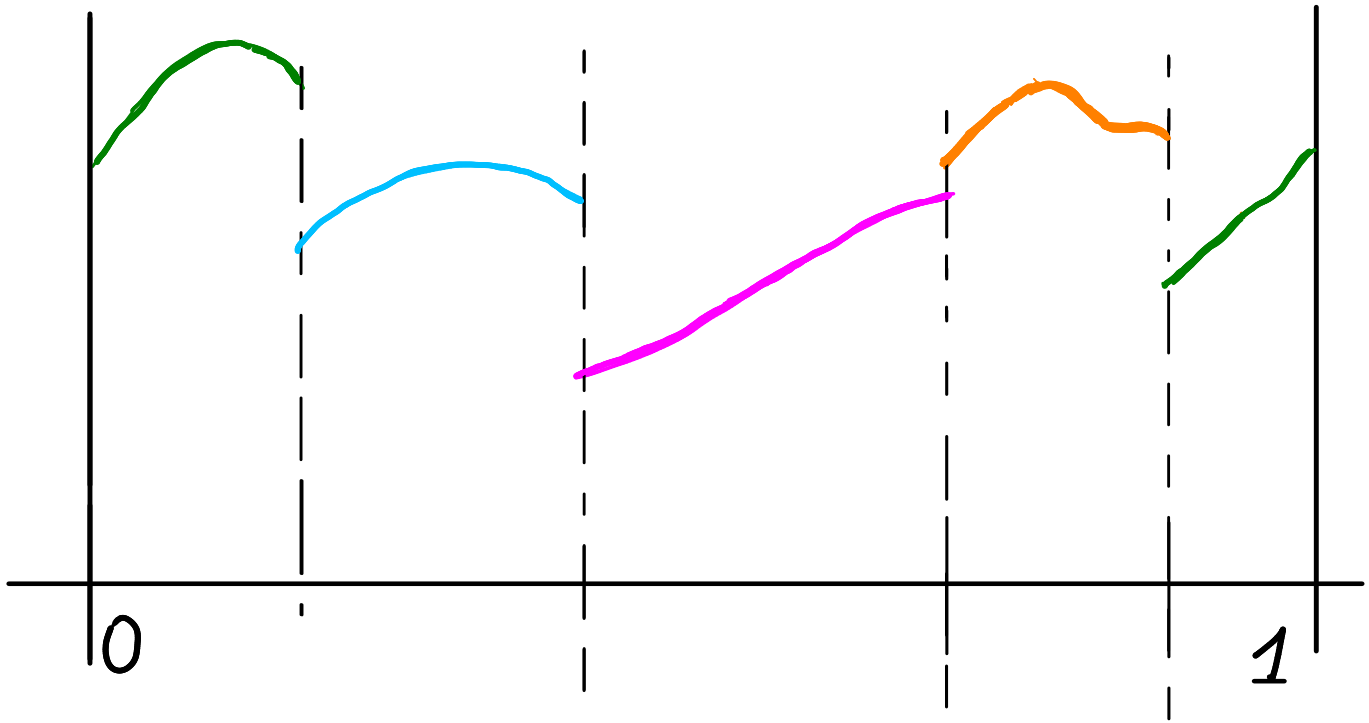

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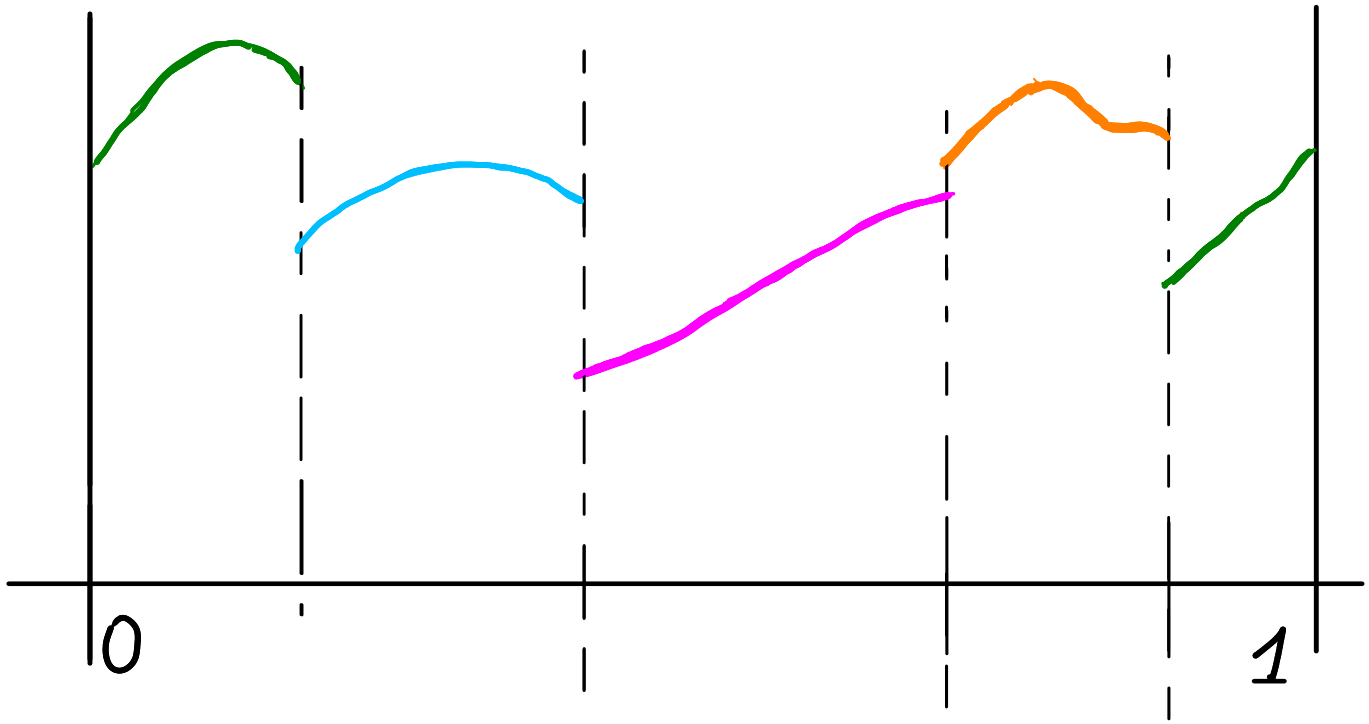


Clearly translation invariant process..

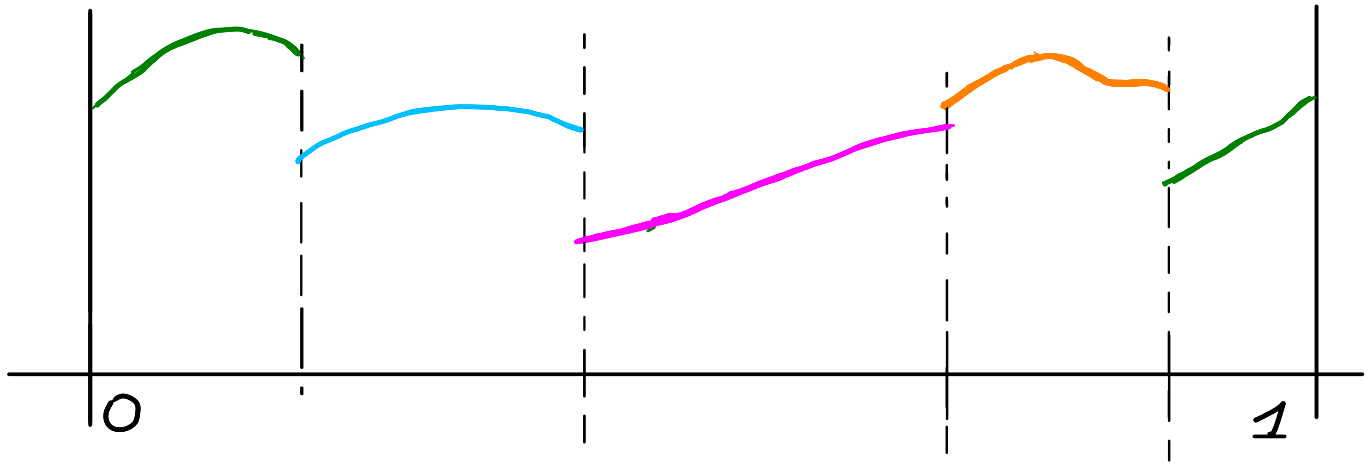
Data analysis: Images

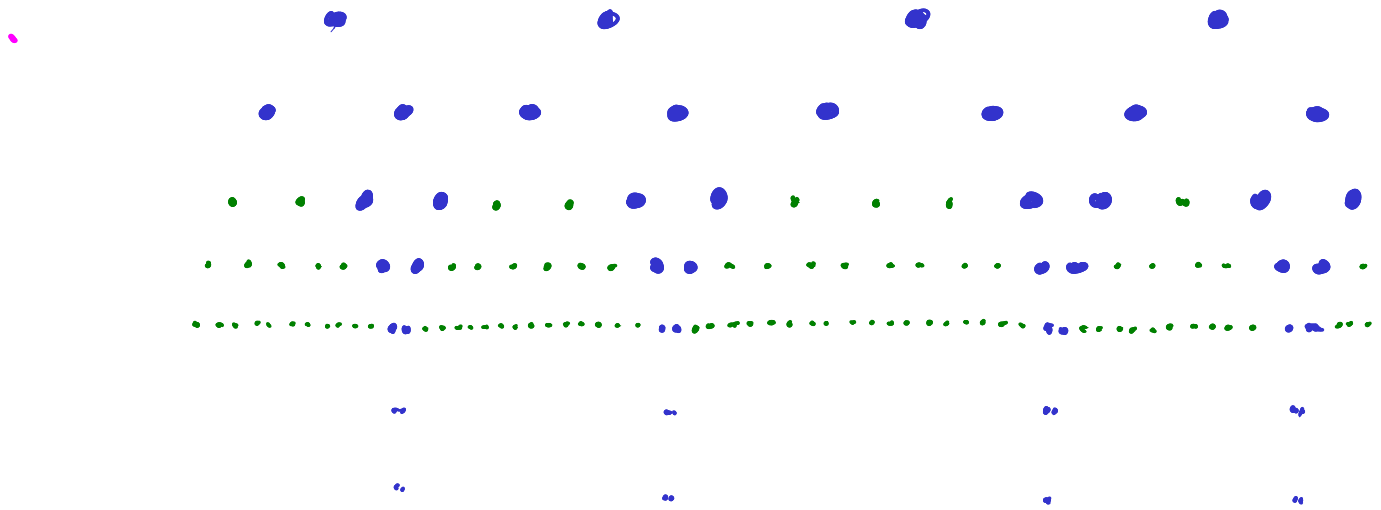
Yet, one can prove that

$$\mathbb{E}\left(\int_{\mathbb{T}} \left| f(t) - \sum_{|n| \leq N} \langle f, e_n \rangle e_n(t) \right|^2 dt \right) \geqslant C \frac{1}{N}$$

But with a wavelet expansion it is very simple to find a strategy that does better ...

Data analysis: Images





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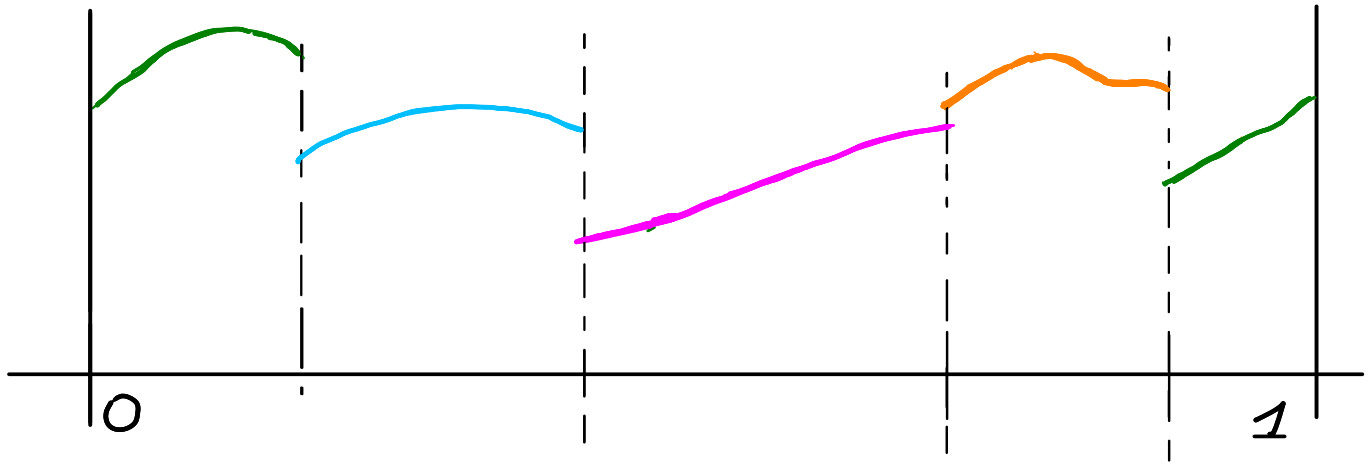

One easily proves that with this strategy,

$$\mathbb{E}\left(\int_{\mathbb{T}} |f(t) - A_{2N+1}(f)(t)|^2 \, dt \right) \leq C N^{-2}$$

where $A_{2N+1}(f)$ is an approximation to f that uses only $2N+1$ coeffs.

Data analysis: Images

But there is an enormous difference.

In 1 case

$$\mathbb{E}\left(\| f - \sum_{\ell=1}^{L} \langle f, \varphi_\ell \rangle \varphi_\ell \|^2 \right)$$

is minimal

In the other case,

$$\mathbb{E}\left(\| f - \sum_{\ell \in \Lambda_L(f)} \langle f, \psi_\ell \rangle \psi_\ell \|^2 \right)$$

is considered, with $\# \Lambda_L(f) = L$

In both cases, L coeffs allowed, but in 2nd case their choice can depend on f.

Data analysis: Images

Linear approximation:

$$\varphi_1, \varphi_2 \dots \varphi_n, \dots \quad \Rightarrow \quad \mathrm{Span}(\varphi_1, \dots \varphi_n) = V_n$$

and study $\mathrm{dist}(f, V_n)$

Nonlinear approximation:

$$\varphi_1, \varphi_2, \dots \varphi_n, \dots$$

$$\Sigma_n = \left\{ \sum_{\ell \in \mathbb{N}} c_\ell \varphi_\ell \; ; \; \#\{\ell; c_\ell \neq 0\} \leq n \right\}$$

now study $\mathrm{dist}(f, \Sigma_n)$.

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Wavelets are a good basis for nonlinear approximation of images, because images have sparse wavelet expansions.

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With hindsight : first example of benefit of sparse expansions.

Why do wavelets have this property ?

Wavelets are connected with beautiful and strong theorems in harmonic analysis

Calderón - Zygmund theory

In fact, wavelets are not even the best basis for 2D - images

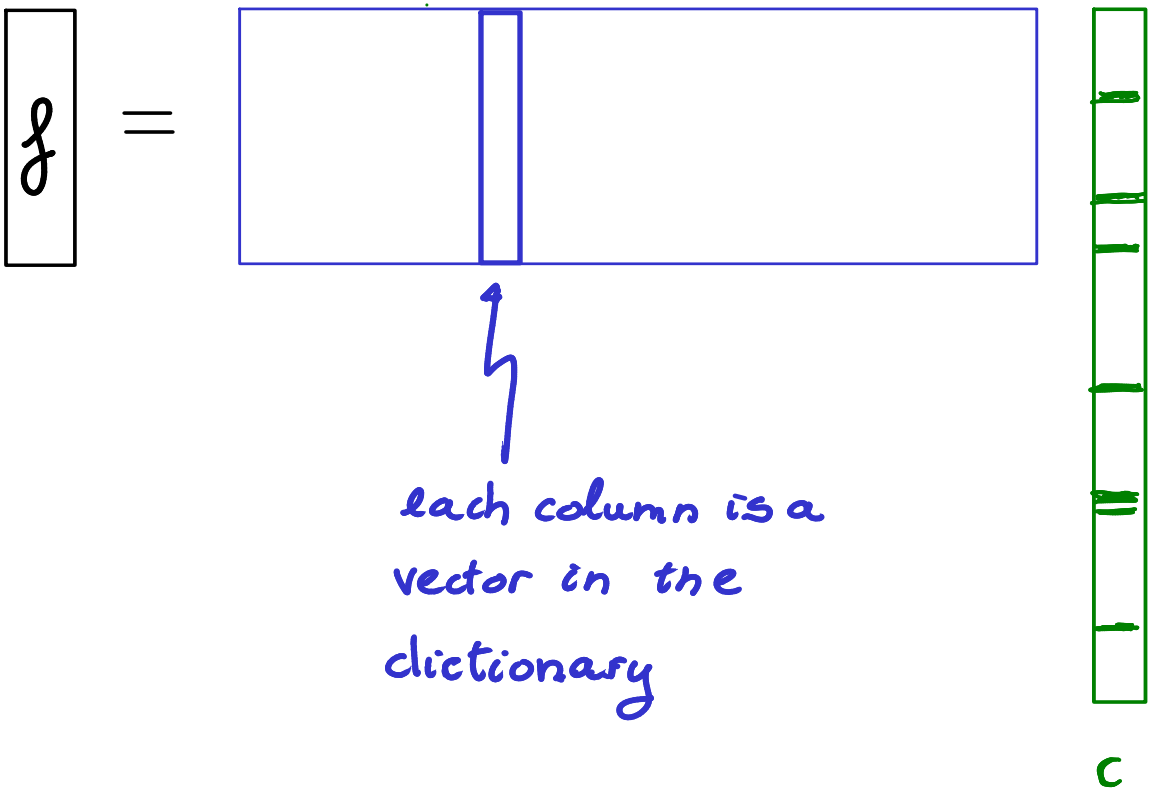
Images really need curvelets (or shearlets)

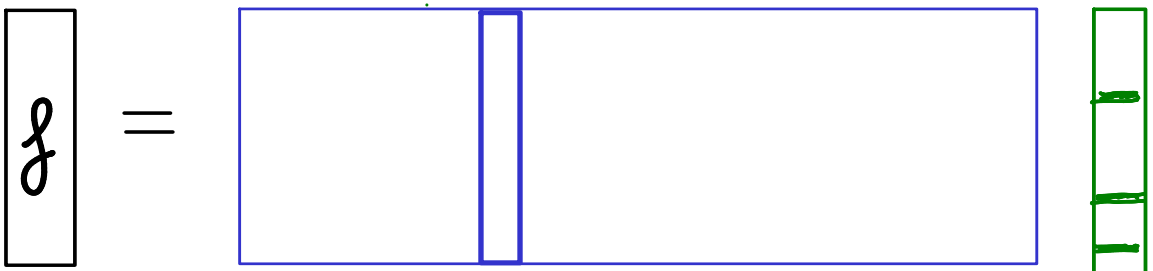
For wavelets, we were lucky : we "guessed" a good basis.

Can we search for a good basis for sparse expansions ?

Find good basis for sparse expansions?

- Search within "dictionaries"
  ↓
  union of many bases.
- Nonlinear (adaptive) singular value decompositions


each column is a
vector in the
dictionary
c



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Same kind of situation as in compressed sensing !

Compressed sensing.

Back to images, for a moment.

Images are sparse when expressed as a combination of wavelets.

For compression applications:

- use fast transform to decompose into wavelets
- retain only the significant coeffs. (identity depends on image)

Why bother first getting all these coeffs? Why not "acquire" image sparsely?

In other words, if we know $x \in \mathbb{R}^N$ is a sparse vector, i.e. $\#\{\ell\,;\, x_\ell \neq 0\} \leq K << N$, can we then determine x by making fewer than N measurements?

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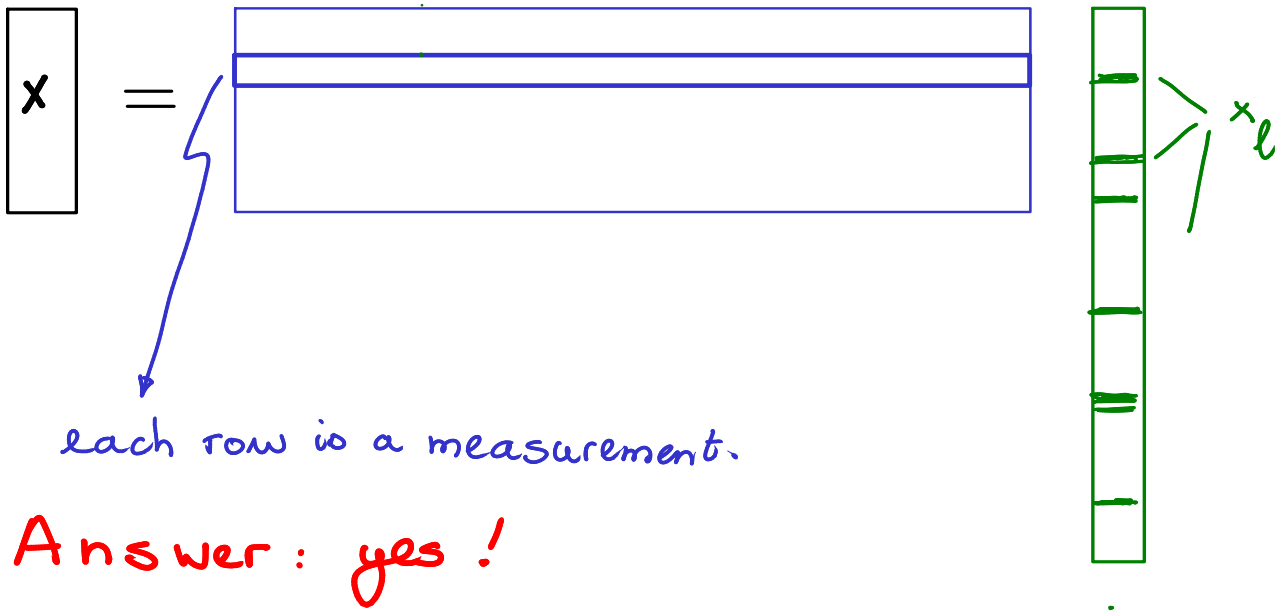


Answer: yes!

Compressive sensing is related to results in theoretical computer science

use Johnson-Lindenstrauss Lemma.

$x_1, \dots x_L$ vectors in V

high-dim. space

$\dim V = D$

Consider projections of x_ℓ on randomly picked d-dim. subspace of V.

Compare $\langle P x_i , P x_j \rangle \frac{D}{d}$ with $\langle x_i, x_j \rangle$

How large should d be for these 2 matrices to be close with high probability?

Basically: $\log L$

This result in CS has had a tremendous impact

- verify that proofs are correct with high probability by "random sampling"
- fast computation algorithms (with small probability of failure).

Fast computations: example.

$f \in \mathbb{C}^N$ N huge

There exists $x \in \mathbb{C}^N$, with only $K << N$ non-zero entries, that is close to f.

$\Rightarrow$ To get a good approximation to f, one needs to take only (non-adaptively!) $O(K \log N)$ random samples of f

and algorithm runs in $O(K \log N)$ time as well.

Finding good ways to represent data.

↳ Knowing (or "believing") that there is a sparse expansion can be exploited to reconstruct from seemingly very insufficient data.

Search in a dictionary
↔ compressed sensing.

Find the dictionary if given a class of objects?



Johnson - Lindenstrauss
Compressed sensing
} dimension reduction.

One last salvo about computation made feasible by "dimension reduction"

↳ comparing surfaces, with applications to biology.

$$d_p(\mathcal{L}, \mathcal{L}') = \inf_{\substack{m : \mathcal{L} \to \mathcal{L}' \\ \text{matching}}} \left[\min_{R \in \text{Euclidean gp.}} \sum_{p \in \mathcal{L}} \| m(p) - Rp \|^2 \right]^{1/2}$$

$$d_p(\mathcal{L}, \mathcal{L}') = \inf_{\substack{m : \mathcal{L} \to \mathcal{L}' \\ \text{matching}}} \left[\min_{R \in \text{Euclidean gp.}} \sum_{p \in \mathcal{L}} \| m(p) - Rp \|^2 \right]^{1/2}$$

$$\mathcal{D}_p(S, S') = \inf_{C : S \to S' \text{ area-preserving}} \left[\min_{R \in \text{Euclidean gp.}} \int_S \| C(x) - Rx \|^2 \, dA_S \right]^{1/}$$

If $\mathcal{D}_p(S, S')$ is small,

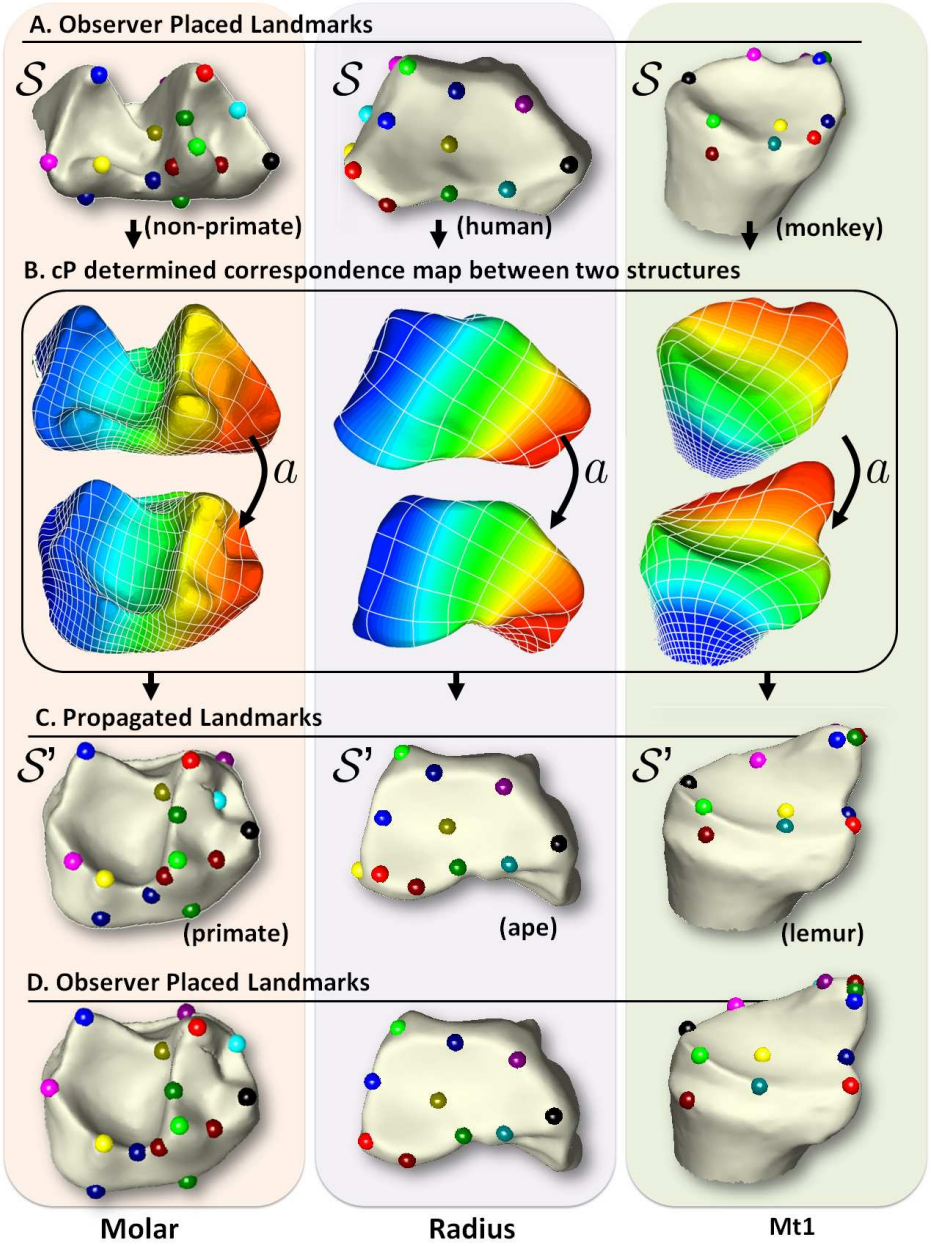
then $\exists$ conformal map $m : S \to S'$

so that

$$\min_R \int_S \| m(x) - Rx \|^2 \, dA_S \leq C \, \mathcal{D}_p(S, S')^{1/2} .$$

↪ use this to compute approx. to $\mathcal{D}_p(S, S')$

by searching "deformations of conformal maps"


A. Observer Placed Landmarks
$\mathcal{S}$
(non-primate)
$\mathcal{S}$
(human)
$\mathcal{S}$
(monkey)
B. cP determined correspondence map between two structures
a
a
a
C. Propagated Landmarks
$\mathcal{S}'$
(primate)
$\mathcal{S}'$
(ape)
$\mathcal{S}'$
(lemur)
D. Observer Placed Landmarks
Molar
Radius
Mt1

With apologies to

A. Cohen
J.P d'Ales
Y. Meyer
R. De Vore
R. Coifman
E. Candès
D. Donoho
J. Romberg
T. Tao
W. Dahmen

L. Carin
A. Gilbert
M. Strauss
R. Calderbank
Y. Lipman
O. Yilmaz
D. Boyer
J. Jernvall

and many, many more ...